A FUNCTIONAL METHOD FOR LINEAR SETS

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ABSTRACT
Kronecker sets for approximation by characters are constructed by an application of Baire's theorem to Banach spaces of differentiable functions.

A compact subset E of $(-\infty,\infty)$ is a *Kronecker set* [2, §5.2] if the exponential functions $e^{i\lambda x}(-\infty < \lambda < \infty)$ are uniformly dense in the continuous complex-valued functions of modulus 1 on E. Wik [3] has constructed Kronecker sets of Hausdorff dimension 1; in fact E can carry a positive measure subject to any prescribed continuity condition weaker than absolute continuity. However, the sets constructed in [3] seem to be very unevenly dispersed; we shall describe a function-space method that necessarily yields Kronecker sets with some degree of symmetry.

Let p be a positive integer and $r_1, r_2, r_3, \cdots$ numbers such that

$$0 < 2r_{n+1} \leqq r_n \leqq 1 \qquad (1 \leqq n < \infty) \tag{1}$$

$$\sup \frac{r_n^p}{r_{n+1}} = \infty. \tag{2}$$

Let Y be the "symmetric set" of numbers $\sum_{n=1}^{\infty} \varepsilon_n r_n$, $\varepsilon_n = \pm 1$ $(1 \leqq n < \infty)$. For some closed interval $I \supseteq Y$, C^p is the Banach space of real functions, p times continuously differentiable in I, normed $\|\phi\| \equiv \max|\phi| + \cdots + \max|\phi^{(p)}|$.

THEOREM: *Except for a subset N of the first category in C^p, each function determines a homomorphism of Y onto a Kronecker set.*

Proof. We use the following fact, whose proof is left to the reader. If $a_1, \cdots a_n$ are complex numbers, and $I_1 \cdots, I_n$ intervals whose mutual distances are $> \delta$, then for some $f \in C^p$

$$f(x) = a_i \text{ for } x \in I_i \qquad 1 \leqq i \leqq n)$$

$$\|f\| \leqq (1 + B\delta^{-p})\max|a_i|$$

for an absolute constant $B \equiv B(p)$.

A function ϕ in C^p belongs to $\sim N$ if each continuous unimodular function h

Received December 20, 1966, and in revised form March, 1967.

on Y can be uniformly approximated by exponentials $e^{i\lambda\phi}(-\infty < \lambda < \infty)$. Because the functions h admit a countable dense subset it is enough to prove that for each *fixed* h, the exceptional set has void interior, since it is evidently of type F_σ.

We denote by $\lambda(\eta)^{-1}(\eta > 0)$ a real number whose nearest neighbors in the sequence $r_1\ r_2, r_3, \cdots$, say $r_n > \lambda(\eta)^{-1} > r_{n+1}$, satisfy the inequalities

$$\eta^2 r_n^p > \eta\lambda(\eta)^{-1} > r_{n+1}; \tag{3}$$

$\lambda(\eta)^{-1}$ and r_n can be made arbitrarily small by (2).

For a number $t > 0$, $\eta_1 \in (0,1)$, and a complex number z of modulus 1 set

$$V(t,z,\eta_1) = \{-\infty < x < \infty, |e^{itx} - z| < \eta_1\}. \tag{4}$$

Every real number is within $2\pi t^{-1}$ of the mid-point of an interval in $V(t,z,\eta_1)$ and each interval has length $\geqq 2\eta_1 t^{-1}$ or $\geqq 2\pi$.

Divide Y into disjoint closed subsets (with mutual distances $\geqq \rho > 0$) on each of which h has oscillation $< \eta_1$. Choose a number $\lambda(\eta)^{-1}$ and further divide Y according to the co-ordinates $\varepsilon_1, \cdots, \varepsilon_n$, where n depends on $\lambda(\eta)$ by the inequality $r_n > \lambda(\eta)^{-1} > r_{n+1}$. The distances between the distinct portions, say E_j $(1 \leqq j \leqq s)$, are at least ρ, or $r_n(1-\eta^2)$, on account of (1) and (3). More exactly, suppose $\sum_{m=1}^{\infty} \varepsilon_m r_m$ and $\sum_{m=1}^{\infty} \varepsilon'_m r_m$ agree up to ε_j with $1 \leqq j < n$. Then

$$\begin{aligned} \left| \sum_{m=1}^{\infty} \varepsilon_m r_m - \sum_{m=1}^{\infty} \varepsilon'_m r_m \right| &\geqq 2r_{j+1} - 2 \sum_{j+2}^{n} r_m - 2 \sum_{n+1}^{\infty} r_m \\ &\geqq 2r_n(2^{n-j-1} - 2^{n-j-2} - \cdots - 1 - 2\eta^2) \\ &= 2r_n(1-2\eta^2) \geqq r_n(p - \eta^2), \text{ for } 1 > 3\eta^2. \end{aligned}$$

By choosing n and η appropriately we make the distances $\geqq \frac{1}{2} r_n$. The length $|E_j| \leqq 2r_{n+1} \leqq 2\eta\lambda(\eta)^{-1}$.

Choose any number z_j in $h(E_j)$ and any number w_j smaller than $2\pi\lambda(\eta)$ such that $w_j + \phi(E_j)$ contains the mid-point of an interval of $V(\lambda(\eta), z_j\ \eta_1))$ (see (4)); the length of the interval $\geqq 2\eta_1\lambda(\eta)^{-1}$ (for large $\lambda(\eta)$). Because the length $|\phi(E_j)| \leqq \|\phi\|\ |E_j| \leqq 2\|\phi\|\eta\lambda(\eta)^{-1}$, when n and η are properly chosen $\phi(E_j) + w_j \subseteq V(\lambda(\eta), z_j, \eta_1)$. This means that

$$|\exp i\lambda(\eta)\ (\phi + w_j) - z_j| < \eta_1 \text{ on } E_j \qquad (1 \leqq j \leqq s),$$

and it is already true that $|h - z_j| < \eta_1$ on E_j $(1 \leqq j \leqq s)$. Now the distance between the E_j's is at least $\frac{1}{2}r_n$; Therefore there is a function ψ in C^p such that $\psi = w_j$ on E_j $(1 \leqq j \leqq s)$ and

$$\begin{aligned} \|\psi\| &\leqq B(1 + 2^p r_n^{-p}) \max|w_j| \leqq B2^{p+1} r_n^{-1}\, 2\pi\lambda(\eta)^{-1} \\ &\leqq 4\pi\eta B \text{ by (3).} \end{aligned}$$

Then $|\exp i\lambda(\eta)(\phi + \psi) - h| < 2\eta_1$ and the proof is complete.

REMARKS. Let L be the Lebesgue function of $Y[1, I]$, so that $dL$is the "natural" probability carried by Y. Then $L(x + \varepsilon) \leqq L(x) + 2^{-m(\varepsilon)}$, whenever $r_{m(\varepsilon)} \geqq \varepsilon$, so that the modulus of countinuity $w_L(\varepsilon) \leqq \varepsilon. (2^{-m(\varepsilon)}/\varepsilon)$. Now let $p = 1$ and observe that (1) and (2) can be attained with $m \log 2 + \log r_m$ converging to $-\infty$ as slowly as we please. But this means that $2^{-m(\varepsilon)}/\varepsilon$ can converge to $+\infty$ as slowly as we please. Thus Y can have positive h-measure for any indicator h such that $\lim_{\varepsilon \to 0} h(\varepsilon)/\varepsilon = +\infty$. Now if we choose $\phi' > 0$, by the theorem proved, $\phi'(Y)$ has positive h-measure. This is Theorem 1 of [3].

REFERENCES

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